

Control of open quantum system dynamics

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We investigate the control resources needed to effect arbitrary quantum dynamics. We show that the ability to perform measurements on a quantum system, combined with the ability to feed back the measurement results via coherent control, allows one to control the system to follow any desired open-system evolution. Such universal control can be achieved, in principle, through the repeated application of only two coherent control operations and a simple “Yes-No” measurement.

03.65.Bz, 05.30.-d, 89.70.+c

Ever since the discovery of quantum mechanics more than a century ago, the problem of controlling quantum systems has been an important experimental issue [1]. A variety of techniques are available for controlling quantum systems, and a detailed theory of quantum control has been developed [2,3]. Methods of geometric and coherent control are particularly powerful for controlling the coherent dynamics of quantum systems [3,4]. In coherent control, a series of semiclassical potentials described by Hamiltonians $\{H_i\}$ are applied to the system. A basic result in coherent control is that, for a finite dimensional quantum system, only a small set of Hamiltonians is required to enact any desired unitary transformation on the system [2,3]. In general, only two different Hamiltonians will suffice to achieve such universal control [5]. Thus, arbitrarily complicated time evolutions can be built up out of simple building blocks.

In this Letter we investigate the corresponding question for open-system, incoherent dynamics. Can an arbitrarily complicated open-system dynamics be built up by applying coherent control together with only a few basic open-system operations? Unlike the case of Hamiltonian dynamics, where complete control can be attained purely in terms of open-loop manipulations as implied above, this turns out to be possible only if closed-loop control (*i.e.*, feedback) is allowed. We show that only one simple coupling to a measurement apparatus, combined with the ability to coherently manipulate the system and feed back the measurement results by the repeated application of a few basic control operations, allows one to enact arbitrary open-system dynamics.

Let us first review the basic result of coherent control theory mentioned above. Consider an n -dimensional quantum system, S , and suppose that one can apply potentials corresponding to a set of Hamiltonians $\{\pm H_i\}$, so that one can apply unitary operators of the form $e^{\pm i H_{i_n} t_n} \dots e^{\pm i H_{i_1} t_1}$. Note that $e^{i H_2 \Delta t} e^{i H_1 \Delta t} e^{-i H_2 \Delta t} e^{-i H_1 \Delta t} = e^{[H_1, H_2] \Delta t^2} + O(\Delta t^3)$, where $[H_1, H_2]$ is the commutator of H_1 and H_2 . Repeating this procedure shows that applying Hamiltonians from the set allows one to enact any coherent evolution of the form $e^{-i A t}$, where A belongs to the algebra generated from the set by commutation. The time evolution is

stroboscopic, matching up to the desired one at discrete intervals, and approximate. But the error of approximation can be made as small as desired by making Δt sufficiently small. Since typically only a small number of Hamiltonians (two, *e.g.*) are required to generate all possible Hamiltonians via commutation, arbitrary coherent transformations can be built up from a small set of simple building blocks. This coherent control technique is open-loop because no information about the state of the system is required to enforce the desired dynamics.

Now consider open-system dynamics. The simplest way to describe an open quantum system is to embed it in a closed quantum system. Let the quantum system and environment be described by a joint density matrix ρ_{SE} . The time evolution of ρ_{SE} is given by a unitary transformation $\rho_{SE} \mapsto U \rho_{SE} U^\dagger$. Before the transformation the system on its own is described by a reduced density matrix $\rho_S = \text{tr}_E \rho_{SE}$. Afterward, its state is $\text{tr}_E U \rho_{SE} U^\dagger$. It is then immediately clear that if one has the ability to enact any coherent transformation on the system and environment taken together, then one can enact any desired open-system transformation of the system on its own [6].

In general, however, one does not have control over the system's environment, but only over the system itself. If the system and the environment are initially uncorrelated, so that $\rho_{SE}(0) = \rho_S \otimes \rho_E$, the time evolution of S can be described by a completely-positive, trace-preserving map $\rho_S \mapsto \sum_k A_k \rho_S A_k^\dagger$, where $\sum_k A_k^\dagger A_k = \mathbb{1}$ and the so-called Kraus operators A_k can be derived from the embedding of the previous paragraph [7]. The most general infinitesimal limit of such a time evolution is given by the Lindblad equation [8]:

$$\begin{aligned} \frac{d\rho_S}{dt} = & -i[H, \rho_S] \\ & - \frac{1}{2} \sum_{i,j} a_{ij} (F_i^\dagger F_j \rho_S + \rho_S F_i^\dagger F_j - 2F_j \rho_S F_i^\dagger), \end{aligned} \quad (1)$$

where $H = H^\dagger$ is the effective system Hamiltonian (the natural Hamiltonian possibly renormalized by a Lamb-shift term), $\{F_i\}$ is a basis for the space of bounded traceless operators on S , and $\mathbf{A} = \mathbf{A}^\dagger = \{a_{ij}\}$ is a positive

semi-definite matrix. After diagonalizing \mathbf{A} , Eq. (1) can be cast in the equivalent canonical form

$$\frac{d\rho_S}{dt} = -i[H, \rho_S] - \frac{1}{2} \sum_k (L_k^\dagger L_k \rho_S + \rho_S L_k^\dagger L_k - 2L_k \rho_S L_k^\dagger), \quad (2)$$

for a set of bounded, traceless *Lindblad operators* $\{L_k\}$ [8]. The Lindblad equation is a Markovian master equation that defines a quantum dynamical *semigroup*: integrating the Lindblad equation over time t defines an open-system transformation Λ_t such that $\Lambda_t \Lambda_s = \Lambda_{t+s}$ for $t, s \geq 0$.

Suppose that, in analogy with the infinitesimal open-loop prescription of the closed-system case, one attempts to build up arbitrary open-system evolutions by applying coherent operations associated with Hamiltonians $\{H_i\}$ together with some set of open-system infinitesimal operations described by Lindblad operators $\{L_k\}$. Clearly, the implementation of the coherent part of the dynamics, corresponding to H , poses no problem. However, one quickly arrives at an impasse while trying to enact the open-system operation corresponding to an arbitrary \mathbf{A} , as the infinitesimal transformations $\Lambda_{\Delta t}$ are *not* invertible in general. This has a drastic effect in determining the set of matrices \mathbf{A} that can be reached by composition of infinitesimal open-loop transformations. If we can apply infinitesimal operations corresponding to two different Lindblad equations specified by \mathbf{A} and \mathbf{A}' , then we can generate an infinitesimal operation corresponding to $\mathbf{A} + \mathbf{A}'$. But it is not in general possible to build up an arbitrary positive semi-definite matrix by adding together a finite set of positive semi-definite matrices. Indeed, it has been shown recently that even in the case of a simple two-state quantum system a *continuous* set of distinct non-unitary open-loop transformations is required to generate an arbitrary Markovian dynamics [9].

Turn now to closed-loop control. In closed-loop control one performs measurements on the system and feeds back the results of these measurements by applying operations that are a function of the measurement results. Although both the system and the measurement apparatus are quantum-mechanical in principle, the feedback operations we consider here involve feeding back *classical*, not quantum information. The case of feeding back quantum information via fully coherent quantum feedback was addressed in [10]. Note that closed-loop control is intrinsically *non*-Markovian in that the feedback loop retains memory of the system's state at previous times [11]. As will now be shown, such non-Markovian feature is essential to allow for the generation of arbitrary open-system dynamics, including arbitrary Markovian Lindblad evolutions, based on a few simple operations.

To analyze closed-loop control, we need to describe quantum measurements. Quantum measurement is a special case of an open-system time evolution. A generalized quantum measurement can always be described

by letting the system to be measured interact with an auxiliary environment (or ancilla) and then performing a conventional von Neumann measurement on the latter [12]. In terms of Kraus operators, a generalized measurement corresponds to a set of operators A_{km} such that $\sum_{km} A_{km}^\dagger A_{km} = \mathbb{1}$. Suppose that the state of S is ρ_S as above. The measurement then gives the result k with probability $p_k = \text{tr} \sum_m A_{km} \rho_S A_{km}^\dagger$, leaving S in the state $\rho_k = \sum_m A_{km} \rho_S A_{km}^\dagger / p_k$. The information that can be gathered about a quantum system is maximized for so-called “pure” measurements [12,13], in which case a single value value of m occurs. We will restrict to pure measurements henceforth.

A particularly interesting class of pure measurements occurs when the Kraus operators are positive (hence Hermitian). Such measurements, which can be realized by coupling the system to a suitable “pointer variable” of the measuring apparatus, are the “least disturbing” measurements that can be effected on S [13]. As such, they supply a natural analogue of von Neumann’s model of projective measurements. Suppose that one can perform such a generalized measurement corresponding to a set of positive operators B_k , and feed back the results by applying a coherent transformation U_k when the k -th outcome is found. The net effect on the system is to apply the transformation

$$\rho_S \mapsto \sum_k U_k B_k \rho_S B_k^\dagger U_k^\dagger. \quad (3)$$

That is, feedback gives rise to an open-system operation corresponding to Kraus operators $A_k = U_k B_k$. But, from the polar decomposition theorem, any operator can be written $A_k = U_k |A_k|$, where $|A_k| = \sqrt{A_k^\dagger A_k}$ is positive and U_k is defined to be $\mathbb{1}$ on the kernel \mathcal{K} of A_k and $A_k |A_k|^{-1}$ on the orthogonal subspace \mathcal{K}^\perp [14]. As a consequence, the ability to make a generalized measurement corresponding to an arbitrary positive operator ($|A_k|$) together with the ability to apply an arbitrary coherent unitary transformation conditioned on the result of that measurement (U_k), allows one to enact an arbitrary open-system operation ($A_k = U_k |A_k|$).

The polar decomposition provides a natural separation of a general quantum operation in terms of a measurement step, followed by a feedback step. This decomposition, which has been implicitly used throughout the history of quantum measurement [1,7,13], has recently proven useful to investigate the trade-off between information and disturbance in quantum feedback control [15]. In our context, since we have assumed the ability to build up arbitrary coherent transformations, we need only to find a way to perform measurements corresponding to arbitrary positive operators in order to enact an arbitrary open-system time evolution.

Consider the simplest possible example of a quantum measurement, a so-called “Yes-No” measurement [7] in which the measurement apparatus M consists of a single

two-level quantum system with states $|0\rangle, |1\rangle$. Assume that $\rho_M = |0\rangle\langle 0|$ initially, and couple the system to the apparatus via an interaction Hamiltonian $H = \gamma X \otimes Y$, where $\gamma > 0$ is a coupling constant, and X, Y are Hermitian operators acting on S, M respectively. We choose X to be a positive operator projecting onto a pure state of S , and let $Y = |0\rangle\langle 1| + |1\rangle\langle 0| = \sigma_x$. The state of the system and apparatus after they interact for a time t is

$$\rho_{SM}(t) = (\cos(\gamma t X) - i \sin(\gamma t X) \otimes \sigma_x) \rho_S \otimes \rho_M \\ (\cos(\gamma t X) + i \sin(\gamma t X) \otimes \sigma_x). \quad (4)$$

Now make a von Neumann measurement of 0,1 on the ancilla. The result 0 occurs with probability $p_0 = \text{tr} \cos^2(\gamma t X) \rho_S$, in which case S is in the state $\rho_0 = \cos(\gamma t X) \rho_S \cos(\gamma t X) / p_0$. The result 1 occurs with probability $p_1 = \text{tr} \sin^2(\gamma t X) \rho_S$, leaving S in the state $\rho_1 = \sin(\gamma t X) \rho_S \sin(\gamma t X) / p_1$. In other words, this Yes-No measurement corresponds to Hermitian Kraus operators $\cos(\gamma t X), \sin(\gamma t X)$.

The ability to perform arbitrary coherent transformations on the system can now be used to transform this simple Yes-No measurement into an arbitrary Yes-No measurement. This can be accomplished by applying an average Hamiltonian technique [16] during the system's coupling to the auxiliary quantum system in the course of the measurement. Before the von Neumann measurement is made on the ancilla, perform the following sequence of rapid coherent transformations on the system: Apply a unitary transformation V_1 , wait for a time Δ_1 , then perform the transformation V_1^\dagger ; now apply a second transformation V_2 , wait for a time Δ_2 , and perform the transformation V_2^\dagger ; *etc.*, for N periods. Assume that the V_i 's are effected on a time scale short compared with the Δ_i , and let $\sum_i \Delta_i = \Delta t$. Repeat $N = t/\Delta t$ times. If N is large, then to lowest order in Δt the net effect of these repeated transformations is to replace X in Eq. (4) by $\bar{X} = \sum_i (\Delta_i / \Delta t) V_i^\dagger X V_i$.

Starting from a given projector X onto a pure state, any positive operator with unit trace can be written in the form \bar{X} for some $V_i, \Delta_i \geq 0$. But with an appropriate choice of the operator \bar{X} , and the parameters γ, t , any pair of positive Kraus operators can be represented in the form $B_0 = \cos(\gamma t \bar{X}), B_1 = \sin(\gamma t \bar{X})$. So our simple measurement procedure together with the ability to perform coherent control translates into the ability to make arbitrary two-outcome minimally disturbing measurements.

Now add feedback. Performing the von Neumann measurement on the ancilla and feeding back the result of the measurement by applying U_0 if the result is 0 and U_1 if the result is 1 gives an open-system time evolution $\rho_S \mapsto A_0 \rho_S A_0^\dagger + A_1 \rho_S A_1^\dagger$, where $A_0 = U_0 \cos(\gamma t \bar{X})$ and $A_1 = U_1 \sin(\gamma t \bar{X})$. But, by polar decomposition, any set of two Kraus operators can be written in this form. As a result, the ability to perform coherent control combined with the ability to perform a single, simple measurement on the system translates into the ability to enact an arbitrary two-operator open-system time evolution.

Generalized measurements with more than two outcomes can be constructed in a straightforward way [7]. For example, if one can make measurements corresponding to arbitrary A_0, A_1 , a measurement corresponding to arbitrary Hermitian B_0, B_1, B_2 , with $B_0^2 + B_1^2 + B_2^2 = \mathbb{1}$, can be enacted as follows. Perform a measurement corresponding to $B_0, B_1' = \sqrt{B_1^2 + B_2^2}$. If the result of this measurement is 0, do nothing. If the result of this measurement is 1, perform a second measurement corresponding to $A_0 = B_1 B_1'^{-1}, A_1 = B_2 B_1'^{-1}$ (note that $B_1'^{-1}$ is well-defined on the set of states that result when the first measurement has outcome 1). This measurement is generally not Hermitian, but the previous paragraph shows that we can do arbitrary generalized measurements with two outcomes. The outcome 0 then corresponds to B_0 , the outcome 10 corresponds to B_1 , and the outcome 11 corresponds to B_2 . Arbitrary measurements with multiple (possibly non-Hermitian) outcomes can be built up by allowing feedback and by following an analogous procedure.

To summarize, the ability to perform a *single* simple measurement on the system, together with the ability to apply coherent control to feed back the measurement results, allows one to enact an arbitrary finite-time open-system evolution $\rho_S \mapsto \sum_k A_k \rho_S A_k^\dagger$.

Now turn to the case of infinitesimal open-system time evolutions governed by the Lindblad equation. To address infinitesimal time evolutions, we can imagine that the interaction between the system and the ancilla representing the measurement apparatus only takes place for a short period of time. Accordingly, the unitary propagator $\exp(-iHt)$ evolving the system and apparatus together can be expanded to second order in time. By writing $H = \gamma X \otimes Y$ as above, one gets

$$\rho_{SM}(t) = \rho_S \otimes \rho_M - i\gamma t (X \rho_S \otimes Y \rho_M - \rho_S X \otimes \rho_M Y) \\ - \frac{\gamma^2 t^2}{2} (X^2 \rho_S \otimes Y^2 \rho_M - 2X \rho_S X \otimes Y \rho_M Y \\ + \rho_S X^2 \otimes \rho_M Y^2). \quad (5)$$

The time evolution for the system on its own is obtained by tracing over the measurement apparatus and taking the small-time limit (*i.e.*, t small but finite [17]). This results in the Lindblad equation for the system,

$$\frac{d\rho_S}{dt} = -ia [X, \rho_S] - b (X^2 \rho_S - 2X \rho_S X + \rho_S X^2), \quad (6)$$

where $a = \gamma \text{tr}_M Y \rho_M = 0$ and $b = (\gamma^2 t / 2) \text{tr}_M Y^2 \rho_M = \gamma^2 t / 2$ for $\rho_M = |0\rangle\langle 0|, Y = \sigma_x$. That is, comparing with (2), the simple coupling above results in a single Lindblad operator $L = \sqrt{2b}X$.

To complete the infinitesimal measurement and feed back the classical information, continue just as in the non-infinitesimal case, by making a von Neumann measurement on the auxiliary system. If the result is 0, do

nothing; if the result is 1, apply the coherent transformation U to S . By using Eq. (5) and tracing over M , the resulting state of the system is

$$\rho_S(t) = \rho_S - \frac{\gamma^2 t^2}{2} (X^2 \rho_S - 2UX\rho_S XU^\dagger + \rho_S X^2). \quad (7)$$

In the small-time limit, this gives rise to a Lindblad equation (2) with $L = \sqrt{2b}UX$. That is, feeding back the result of the infinitesimal measurement yields a Lindblad equation with a single Lindblad operator proportional to UX , where X is a unit-trace positive operator and U is of our choosing.

The average Hamiltonian techniques described above can now be used to construct an infinitesimal measurement corresponding to an arbitrary positive operator \bar{X} . Here, care must be taken to insure that the sequence of average Hamiltonian operations V_i can be performed within the small “coarse-graining” time t above. Note that what is important in the derivation of the Lindblad equation (6) is not that t be small but rather that γt be small: accordingly, even though the relevant control time scale Δt is not vanishingly small, γ can always be made sufficiently small that the averaging operations can be fit within a time t for which the derivation holds. Feeding back the results of the measurement allows one to enact a Lindblad equation governed by an arbitrary single Lindblad operator $L = \sqrt{2b}U\bar{X}$.

The same conclusion could have been also established directly from the ability to implement arbitrary two-operator open-system evolutions. In fact, in the small-time limit discussed above, the quantum operation specified by $A_0 = U_0 \cos(\gamma t \bar{X}) \simeq \mathbb{1} - (\gamma^2 t^2/2) \bar{X}^2$, $A_1 = U_1 \sin(\gamma t \bar{X}) \simeq \gamma t U_1 \bar{X}$, with $U_0 = \mathbb{1}, U_1 = U$, is formally equivalent to the Lindblad equation (2) with $L = \sqrt{2b}U\bar{X}$. In terms of the matrix \mathbf{A} appearing in (1), the ability to generate an arbitrary Lindblad operator $L = \sum_i c_i F_i$, $c_i \in \mathbb{C}$, translates into the ability to generate any rank-one matrix $\mathbf{A} = \{c_i c_j^*\}$ [9]. As noted above, Lindblad equations with multiple L_k can be enacted by performing the operations corresponding to each of the L_k in succession. So by making a simple infinitesimal measurement, manipulating it by average Hamiltonian techniques, and coherently feeding back the results of the measurement, one can enact in principle an arbitrary Lindblad equation. The enactment is scaled in time, stroboscopic, and approximate, but the scaled time interval of the stroboscopic evolution and the error in the approximation can be made as small as possible by increasing the number of control operations per step.

In conclusion, coherent control combined with feedback allows one to enact an arbitrary open quantum system dynamics. In contrast to the case of closed quantum systems, in which an arbitrary Hamiltonian time evolution can be enacted using only a few open-loop operations, the open-loop control of open-system dynamics requires in general an infinite repertoire of independent operations. But closing the loop and feeding back

the results of the measurements allows any desired open-system transformation to be enacted using only the coherent tools required for Hamiltonian control together with a single, simple quantum measurement. The ability to implement a similar control strategy in the appropriate small-time limit allows the construction of any desired continuous-time Markovian evolution described by a Lindblad master equation as well.

This work was supported by DARPA/ARO under the QUIC initiative.

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